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CONVERSES TO THE Ω-STABILITY AND INVARIANT LAMINATION THEOREMS

BY

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ABSTRACT. In 1967 Smale proved that for diffeomorphisms on closed smooth manifolds, Axiom A and no cycles are sufficient conditions for Ω-stability and asserted the analogous theorem for vectorfields. Pugh and Shub have supplied a proof of the latter. Since then a major problem in dynamical systems has been Smale's conjecture that Axiom A (resp. A′) and no cycles are necessary as well as sufficient for Ω-stability of diffeomorphisms (resp. vectorfields). Franks and Guckenheimer have worked on the diffeomorphism problem by strengthening the definition of Ω-stable diffeomorphisms. In this paper it will be shown that an analogous strengthening of Ω-stable vectorfields forces Smale's conditions to be necessary. The major result of this paper is the following THEOREM. *If* (Λ, L) *is a compact laminated set,* N *is a normal bundle to the lamination, and* f *is an absolutely and differentiably* L*-stable diffeomorphism of a closed smooth manifold then* $(\mathrm{id} - \overline{f_{\#}})\colon C^0(N) \longrightarrow C^0(N)$ *is surjective.* If the lamination is just a compact submanifold, the theorem is already new. When applied to flows, this theorem gives the above result on Ω-stable vectorfields.

1. **Introduction.** The object of this work is to show that, for vectorfields on closed smooth manifolds, Smale's Axiom A′ and no cycles are consequences of certain stability hypotheses. In [16] Smale proved that for diffeomorphisms, Axiom A and no cycles are sufficient conditions for Ω-stability and asserted the analogous theorem for vectorfields. In [13] Pugh and Shub supplied a proof of the latter. Since then a major problem in dynamical systems has been Smale's conjecture that Axiom A (resp. A′) and no cycles are necessary as well as sufficient for Ω-stability of diffeomorphisms (resp. vectorfields).

More recently, John Franks has shown that strengthening the definition of Ω-stable diffeomorphisms (by adding two regularity conditions to the conjugacy)

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makes Smale's conditions necessary as well as sufficient [4]. John Guckenheimer has improved this by dropping one of the regularity conditions [5]. But, as of now, the original problem is not solved for diffeomorphisms.

In this paper it will be shown that adding analogous requirements to Ω-stable vectorfields again forces Smale's conditions to be necessary. Moreover, some of the added requirements are shown to be consequences of Smale's conditions. The remaining ones are conjectured to follow as well and partial results in that direction are given.

The major result of this paper, a transversality type theorem, is

(1.1) THEOREM. *If* (Λ, L) *is a compact laminated set and* f *is an absolutely and differentiably* L*-stable diffeomorphism of a closed smooth manifold* M, *then* $(\mathrm{id} - \overline{f_{\#}})\colon C^0(N) \longrightarrow C^0(N)$ *is surjective.*

This theorem is a partial converse of the invariant lamination theorem of [7]. When the lamination reduces to a single fixed point, (1.1) implies the known result that stable fixed points are transversal [4]. In case the lamination is a single compact submanifold the theorem is new. Mañé [21] has obtained a related result.

In §2 definitions are given and (1.1) is proved. In §3 this is applied to flows to show the necessity of Axiom A′ and no cycles for a strengthened version of Ω-stability.

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2. **Definitions and proof of (1.1).** Let M be a closed smooth manifold.

A C^r *k-lamination* of a subset $\Lambda \subset M$ is a function $L\colon \Lambda \longrightarrow 2^\Lambda$ ($L(x)$ is written L_x and is called the *laminum through* x) satisfying for all x and y in Λ

(1) $x \in L_x$.

(2) There exist a connected k-manifold V_x and an injective C^r immersion $i_x\colon V_x \longrightarrow M$ with $i_x(V_x) = L_x$.

(3) Either $L_x \cap L_y = \emptyset$ or $L_x = L_y$.

(4) There exist U, a neighborhood of $x \in \Lambda$, and a continuous map $H\colon U \longrightarrow C^r(D^k, M)$ such that $H(x)\colon (D^k, 0) \longrightarrow (L_x, x)$ is an imbedding of the k disk. H is called a *local chart for* L.

Such a Λ is called laminated. The above is a generalization of a foliation which can be viewed as a lamination where $\Lambda = M$ [20].

Let $P_k(M)$ be the Grassmann bundle of k planes tangent to M. The map $TL\colon \Lambda \longrightarrow P_k(M)$ defined by: $TL(x)$ (in [7] this is written TL_x but not so here) is the k plane tangent to L_x at x. The *tangent space to the laminated set* Λ (which is ambiguously denoted TL rather than $T\Lambda$ to emphasize the dependence on the lamination itself) is $\bigcup_{x\in\Lambda} TL(x)$. TL_x or $T(L_x)$ is used to denote $\bigcup_{y\in L_x} TL(y)$. TL is a C^0 subbundle of $TM|_\Lambda$ and TL_x is a C^1 subbundle of $TM|_{L_x}$.

A diffeomorphism f of M is called an *automorphism of the lamination* L if Λ is f-invariant and f takes lamina onto lamina, i.e., $f \circ L = L \circ f$.

An *isomorphism of laminations* $h: (\Lambda_1, L_1) \longrightarrow (\Lambda_2, L_2)$ is a homeomorphism $h: \Lambda_1 \longrightarrow \Lambda_2$ satisfying $L_2 \circ h = h \circ L_1$, where L_i is a lamination of Λ_i, i.e., h takes lamina onto lamina.

Let L be a lamination of the compact set Λ and let f be an automorphism of L. f is called *strongly L-stable* if there exist N_f, an open neighborhood of $\mathrm{id} \in \mathrm{Diff}^1(M)$, and a map $h: N_f \longrightarrow C^0(\Lambda, M)$ such that

(1) $h(k)(\Lambda)$ is a laminated set, (Λ', L'), $g = k \circ f$ is an automorphism of (Λ', L'), and $h(k): (\Lambda, L) \longrightarrow (\Lambda', L')$ is an isomorphism of laminations.

(2) $h(\mathrm{id}) = i: \Lambda \longrightarrow M$ the inclusion (in this case $L' = L$) and h is continuous at id.

(3) The local representative of h, $\theta: (C^1(TM), 0) \longrightarrow (C^0(TM|_\Lambda), 0)$, when restricted to $C^1(TM, N) \subset C^1(TM)$, actually takes on values in $C^0(N) \subset C^0(TM|_\Lambda)$.

Roughly, (3) means that, for perturbations of f, h gives the new lamination as a section of a normal bundle over the old lamination. Precisely, note that $C^1(M, M)$ and $C^0(\Lambda, M)$ are smooth (Banach) manifolds–$C^1(M, M)$ modeled on $C^1(TM)$, C^1 sections of TM, and $C^0(\Lambda, M)$ modeled on $C^0(TM|_\Lambda)$. As $N_f \subset C^1(M, M)$ is open, it is also modeled on $C^1(TM)$. For f, g, and h any functions having suitable domains and ranges, $\omega_f(g) = f \circ g$ and $\alpha_f(h) = h \circ f$. It is well known [3] that $\omega_{\exp}$ maps a neighborhood of the zero section, $0 \in C^1(TM)$, diffeomorphically onto a neighborhood of $\mathrm{id} \in C^1(M, M)$ and also maps a neighborhood of $0 \in C^0(TM|_\Lambda)$, diffeomorphically onto a neighborhood of the inclusion, $i \in C^0(\Lambda, M)$. With these trivializations the local representative for h is the map θ defined by the following diagram:

$$\begin{array}{ccc}
\mathrm{Diff}^1(M) & \xrightarrow{\;h\;} & C^0(\Lambda, M) \\
\uparrow \omega_{\exp} & & \uparrow \omega_{\exp} \\
C^1(TM) & \xrightarrow{\;\theta\;} & C^0(TM|_\Lambda)
\end{array}$$

where the above maps are defined only on neighborhoods of id and 0. The N mentioned in (3) is a smooth normal bundle to TL, i.e., a continuous subbundle of $TM|_\Lambda$ which, when restricted to a laminum L_x, is a smooth complement to TL_x. N is constructed later. $C^1(TM, N) \subset C^1(TM)$ is the subspace consisting of sections that take Λ into N. The assertion of (3) is that, for such sections s, $\theta(s)$ is a section of N.

Fix a Riemannian metric on M and denote its induced Finsler by $\| \ \|$. For any section s of any subbundle of TM, define its C^0 *norm* by $\|s\|_0 = \sup_x \|s(x)\|$. If $V \subset M$ is a C^r submanifold $(r > 0)$, $E \longrightarrow B$ is a C^r subbundle of TM, and $s \in C^r(E)$; the C^r *norm* of s, $\|s\|_r$, is the maximum of the C^0 norms of s and its first r derivatives (see [1] for details). If (Λ, L) is a C^r laminated set, $E \longrightarrow \Lambda$ is a C^0 subbundle of TM that is C^r over each lamina, and s is a C^0 section of E which is C^r when restricted to each lamina; the C^r *norm* of s, $\|s\|_r$, is $\sup_{x \in \Lambda} \|s|_{L_x}\|_r$ and the space of all such s having finite norm is denoted by $C^{r,0}(E)$.

A strongly L-stable diffeomorphism f is *differentiably* L-*stable* if the following hold:

(1) When $C^1(TM)$ is given the C^1 norm and $C^0(TM|_\Lambda)$ is given the C^0 norm, h is differentiable at $\mathrm{id} \in N_f$.

(2) For $s \in C^1(TM, N)$, $\theta(s) \in C^{1,0}(N)$ and

$$\theta|_{C^1(TM,N)}: C^1(TM, N) \longrightarrow C^{1,0}(N)$$

is continuous at 0 when each side is given the C^1 norm.

A strongly L-stable diffeomorphism f is *absolutely* L-*stable* if $\theta|_{C^1(TM,N)}$ is locally lipschitz in the C^0 sense, at $0 \in C^1(TM, N)$, i.e., if there exist positive constants K and λ such that $\|\theta(s)\|_0 \leqslant K\|s\|_0$ for all $s \in C^1(TM, N)$ with $\|s\|_1 \leqslant \lambda$.

Any diffeomorphism $f: M \longrightarrow M$ induces a linear isomorphism $f_\#: C^0(TM) \longrightarrow C^0(TM)$ defined by $f_\#(s) = Tf \circ s \circ f^{-1}$. If f is an automorphism of a laminated set (Λ, L), $f_\#$ restricts to isomorphisms $f_\#: C^0(TL) \longrightarrow C^0(TL)$ and $f_\#: C^0(TM|_\Lambda) \longrightarrow C^0(TM|_\Lambda)$. If N is the previously mentioned normal bundle to TL, the exact sequence

$$0 \longrightarrow C^0(TL) \longrightarrow C^0(TM|_\Lambda) \longrightarrow C^0(N) \longrightarrow 0$$

and the above restrictions of $f_\#$ induce a cokernel isomorphism, $\overline{f_\#}: C^0(N) \longrightarrow C^0(N)$, commuting the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^0(TL) & \longrightarrow & C^0(TM|_\Lambda) & \longrightarrow & C^0(N) & \longrightarrow & 0 \\ & & \downarrow f_\# & & \downarrow f_\# & & \downarrow \overline{f_\#} & & \\ 0 & \longrightarrow & C^0(TL) & \longrightarrow & C^0(TM|_\Lambda) & \longrightarrow & C^0(N) & \longrightarrow & 0 \end{array}$$

If f is C^r, all the above maps are isomorphisms of the C^{r-1} sections and, using C^{r-1} sections (the last terms become $C^{r-1,0}(N)$), the sequences remain exact and the diagrams remain commutative.

To define N, first choose N_0, a C^0 complement to TL and extend it to $N_1 \longrightarrow U$, a C^0 subbundle of $TM|_U$, with U a neighborhood of Λ (see [6] for details). Next approximate N_1 by N_2, a C^1 subbundle of $TM|_U$, which is such a good C^0 approximation that the openness of transversal intersections and the compactness of Λ guarantee that, for all $x \in \Lambda$,

$$N_2|_x \oplus TL(x) = TM|_x.$$

Finally, let $N = N_2|_\Lambda$. Using standard techniques, one easily proves

(2.1) LEMMA. *$C^1(TM, N)$ is C^0 dense in $C^0(TM, N)$, the subspace of $C^0(TM)$ consisting of sections that take Λ into N.*

PROOF OF (1.1). Choose U a neighborhood of $i \in C^0(\Lambda, M)$ trivialized by $\omega_{\exp}$ and, if necessary, shrink N_f so that it is trivialized by $\omega_{\exp}$. Define $B: N_f \times U \longrightarrow U$ via $B(k, h) = k \circ f \circ h \circ \hat{f}^{-1} - h$ where $\hat{}$ signifies restriction to Λ and the subtraction is performed in $C^0(\Lambda, M)$ via the trivialization. Let $\bar{B}$ represent B, i.e.

$$\begin{array}{ccc} N_f \times U & \xrightarrow{\quad B \quad} & U \\ \Big\uparrow {\scriptstyle \omega_{\exp} \times \omega_{\exp}} & & \Big\uparrow {\scriptstyle \omega_{\exp}} \\ C^1(TM) \times C^0(TM|_\Lambda) & \xrightarrow{\quad \bar{B} \quad} & C^0(TM|_\Lambda) \end{array}$$

(2.2) LEMMA. *$D\bar{B}(0, 0)(a, b) = R(a) - (\mathrm{id} - f_\#)(b)$ where $R: C^1(TM) \longrightarrow C^0(TM|_\Lambda)$ is the restriction.*

PROOF OF (2.2). To compute $D_1\bar{B}(0, 0)$, the first partial, we calculate the tangent map of

$$(\omega_{\exp})(a) \longmapsto B((\omega_{\exp})(a), i)$$
$$= (\omega_{\exp})(a) \circ f \circ i \circ \hat{f}^{-1} - i = \widehat{(\omega_{\exp})(a)} = R((\omega_{\exp})(a)).$$

Since R is linear, $D_1\bar{B}(0, 0) = R$. For $D_2\bar{B}(0, 0)$ we need the tangent of

$$(\omega_{\exp})(b) \longmapsto B(\mathrm{id}, (\omega_{\exp})(b))$$
$$= \mathrm{id} \circ f \circ (\omega_{\exp})(b) \circ \hat{f}^{-1} - (\omega_{\exp})(b)$$
$$= (\omega_f \circ \alpha_{\hat{f}-1} - \mathrm{id})((\omega_{\exp})(b)).$$

Now apply the α and ω lemmas [10] to conclude that

$$D_2\bar{B}(0, 0) = (\omega_{Tf} \circ \alpha_{\hat{f}-1} - \mathrm{id}) = (f_\# - \mathrm{id}).$$

The lemma follows easily.

Returning to (1.1), define $A: N_f \longrightarrow N_f \times U$ by $A(k) = (k, h(k))$ where h is the "conjugacy selector" that exists since f is strongly L-stable. Again using $\omega_{\exp}$ let $\bar{A}$ be the local representative of A. Clearly $D\bar{A}(0) = (\mathrm{id}, D\theta(0))$. By the chain rule $D(C)(0) = R - (\mathrm{id} - f_{\#}) \circ D\theta(0)$ where $C = \bar{B} \circ \bar{A}$. So we have

$$\begin{array}{ccc}
\mathrm{Diff}^1(M) \supset N_f & \xrightarrow[B \circ A]{k \longmapsto k \circ f \circ h(k) \circ f^{-1} - h(k)} & U \subset C^0(\Lambda, M) \\
\uparrow{\scriptstyle \omega_{\exp}} & & \uparrow{\scriptstyle \omega_{\exp}} \\
C^1(TM) & \xrightarrow[s \longmapsto (\omega_{\exp})^{-1}(\exp \circ s \circ f \circ \exp \circ \theta(s) \circ f^{-1}) - \theta(s)]{C} & C^0(TM|_\Lambda)
\end{array}$$

Now consider

$$C^1(TM, N) \xrightarrow{i} C^1(TM) \xrightarrow{C} C^0(TM|_\Lambda) \xrightarrow{P_N} C^0(N).$$

As i, the inclusion, and P_N, the projection, are linear,

$$\begin{aligned} D(P_N \circ C \circ i)(0) &= P_N \circ D(C)(0) \circ i \\ &= P_N \circ R \circ i - P_N \circ (\mathrm{id} - f_{\#}) \circ D\theta(0) \circ i. \end{aligned}$$

By condition (3) of strong L-stability, $\mathrm{image}(\theta \circ i) \subset C^0(N)$. Hence

$$\mathrm{image}(D\theta(0) \circ i) \subset C^0(N).$$

Clearly $\mathrm{image}(R \circ i) \subset C^0(N)$ so

$$D(P_N \circ C \circ i)(0) = \{R - (\mathrm{id} - \overline{f_{\#}}) \circ D\theta(0)\} \circ i. \tag{2.3}$$

(2.4) LEMMA. *$D(P_N \circ C \circ i)(0)$ is the zero map.*

Whereas in [4] there is no trouble proving the corresponding assertion, (2.4) is a technical analytic lemma. The relevant estimates will be given following the proof of (1.1).

By (2.3) and (2.4), $R = (\mathrm{id} - \overline{f_{\#}}) \circ D\theta(0)$ on $C^1(TM, N)$. R extends to a continuous restriction $R': C^0(TM, N) \longrightarrow C^0(N)$.

(2.5) LEMMA. *R' is surjective.*

PROOF OF (2.5). Choose $s \in C^0(N)$ and extend it to $s_1 \in C^0(TM|_U)$ with U a neighborhood of Λ. Now choose a C^0 bump function $j: M \longrightarrow R$ with support in U and satisfying $j|_\Lambda = 1$. $R'(j \cdot s_1) = s$.

(2.6) LEMMA (CF. [4]). *$\|D\theta(0)(s)\|_0 \leq (K+1)\|s\|_0$ for all s in $C^1(TM, N)$ where K is the constant guaranteed to exist by the absolute L-stability of f.*

PROOF OF (2.6). By Taylor's theorem.

(2.7) Given $\epsilon > 0$ there exists $\delta > 0$ such that, for $t \in C^1(TM, N)$ with $\|t\|_1 \leqslant \delta$, $\|\theta(t) - D\theta(0)(t)\|_0 \leqslant \epsilon \|t\|_1$.

Pick a nonzero $s \in C^1(TM, N)$ and let $\epsilon = \|s\|_0/\|s\|_1$. Let δ be as in (2.7) and choose $r > 0$ so that $\|rs\|_1 < \delta$. By absolute stability

$$\begin{aligned} r\|D\theta(0)(s)\|_0 &= \|D\theta(0)(rs)\|_0 \leqslant \|\theta(rs)\|_0 + \|\theta(rs) - D\theta(0)(rs)\|_0 \\ &\leqslant K\|rs\|_0 + \epsilon\|rs\|_1 \leqslant Kr\|s\|_0 + (\|s\|_0/\|s\|_1)\cdot r\|s\|_1 \\ &\leqslant r(K+1)\|s\|_0. \end{aligned}$$

Returning to (1.1), apply (2.6) to conclude that $D\theta(0)\colon C^1(TM, N) \longrightarrow C^0(N)$ is continuous if each side is given the C^0 topology. By (2.1), $D\theta(0)$ extends to a continuous map $E\colon C^0(TM, N) \longrightarrow C^0(N)$. Thus we have two continuous maps, R' and $(\mathrm{id} - \overline{f_\#}) \circ E$, defined on $C^0(TM, N)$, which are equal on $C^1(TM, N)$. Again using (2.1), we see that $R' = (\mathrm{id} - \overline{f_\#}) \circ E$ on $C^0(TM, N)$. Since, by (2.5), R' is surjective, so is $\mathrm{id} - \overline{f_\#}$ completing (1.1).

PROOF OF (2.4). $D(P_N \circ C \circ i)(0)$ exists and $P_N C(0) = 0$. Thus, for $s \in C^1(TM, N)$, as $\|s\|_1 \longrightarrow 0$,

$$\|P_N \circ C(s) - D(P_N \circ C \circ i)(0)(s)\|_0/\|s\|_1 \longrightarrow 0.$$

In particular, for $x \in \Lambda$, as $\|s\|_1 \longrightarrow 0$,

$$\|P_N \circ C(s)(x) - D(P_N \circ C \circ i)(0)(s)(x)\|/\|s\|_1 \longrightarrow 0. \tag{2.8}$$

(2.9) CLAIM. To prove (2.4) it suffices to show that, for $x \in \Lambda$ and $s \in C^1(TM, N)$,

$$\|P_N \circ C(s)(x)\|/\|s\|_1 \longrightarrow 0 \quad \text{as} \quad \|s\|_1 \longrightarrow 0. \tag{2.10}$$

PROOF OF (2.9). Assume, on the contrary, that (2.10) holds but there exist x and s such that $D(P_N \circ C \circ i)(0)(s)(x) = v \neq 0$. Let $a > 0$.

$$\begin{aligned} &\frac{\|P_N \circ C(as)(x) - D(P_N \circ C \circ i)(0)(as)(x)\|}{\|as\|_1} \\ &\qquad = \frac{\|P_N \circ C(as)(x) - aD(P_N \circ C \circ i)(0)(s)(x)\|}{\|as\|_1} \\ &\qquad \geqslant \frac{a\|D(P_N \circ C \circ i)(0)(s)(x)\|}{a\|s\|_1} - \frac{\|P_N \circ C(as)(x)\|}{\|as\|_1} \\ &\qquad = \frac{\|v\|}{\|s\|_1} - \frac{\|P_N \circ C(as)(x)\|}{\|as\|_1}. \end{aligned} \tag{2.11}$$

Now let $a \longrightarrow 0$. By (2.8), (2.11) should $\longrightarrow 0$. It does not, however, since $\|v\|/\|s\|_1 > 0$ and, by hypothesis (2.10) $\|P_N \circ C(as)(x)\|/\|as\| \longrightarrow 0$.

Again returning to (1.1), let $k = \exp \circ s$, $g = k \circ f$, and $h_g = h(k) = \exp \circ \theta(s)$. Thus $C(s) = (\omega_{\exp})^{-1}(g \circ h_g \circ f^{-1}) - (\omega_{\exp})^{-1}(h_g)$. By (2.9) we may fix x for all calculations. Choose a compact connected subset of $L_{f(x)}$ containing $f(x)$ in its interior and, by abuse of notation, refer to this piece as $L_{f(x)}$. By compactness exp imbeds (a neighborhood of the zero section of) $N|_{L_{f(x)}}$ into M as a tubular neighborhood of $L_{f(x)}$. Since f is differentiably L-stable, for small $\|s\|_1$, $h_g(L_x)$ is C^1 near L_x and, therefore, can be considered as the image of a section of the imbedded normal bundle whose projection map is denoted by p.

(2.12) ASSERTION. *For small* $\|s\|_1$, $g \circ h_g(x) = h_g \circ p \circ g \circ h_g(x)$.

PROOF OF (2.12). Since $\|s\|_1$ is small, $y = g \circ h_g(x)$ lies in the imbedded normal bundle. As h_g is the L-conjugacy, $y \in h_g(L_{f(x)})$. But so is $h_g \circ p(y)$. Since both lie in the fiber over $p(y)$, they must be equal.

The Riemannian metric on TM induces many others: d is the induced metric on M, d_0 is the C^0 metric on $C^0(M, M)$, d_1^x is the C^1 metric on $C^1(L_x, M)$, and d_1 is the $C^{1,0}$ metric on

$$C^{1,0}(\Lambda, M) = \{h \in C^0(\Lambda, M) | \text{ for all } x \in \Lambda, h \in C^1(L_x, M)\}$$

defined by $d_1(h) = \sup_x d_1^x(h|_{L_x})$.

(2.13) ASSERTION. *There exists* $\bar{K} > 0$ *such that, for small* $\|s\|_1$, $d(p \circ g \circ h_g(x), p \circ f(x)) \leqslant \bar{K}\|s\|_1$.

PROOF OF (2.13). Since h is differentiable at id (f is differentiably L-stable), there exists K_1 such that, for small $\|s\|_1$, $d_0(h_g, \text{id}) \leqslant K_1\|s\|_1$. Similarly there exists K_2 such that, for y near x, $d(f(y), f(x)) \leqslant K_2 d(y, x)$. Since $s \longmapsto k$ is differentiable at 0, there exists K_3 satisfying, for small $\|s\|_1$, $d(k, \text{id}) \leqslant K_3\|s\|_1$. Finally, p is differentiable at $f(x)$. So, there exists K_4 such that, for w near $f(x)$, $d(p(w), p \circ f(x)) \leqslant K_4 d(f(x), w)$. Therefore, for small $\|s\|_1$,

$$\begin{aligned} d(p \circ g \circ h_g(x), p \circ f(x)) &\leqslant K_4 d(g \circ h_g(x), f(x)) \\ &= K_4 d(k \circ f \circ h_g(x), f(x)) \\ &\leqslant K_4 \{d(k \circ f \circ h_g(x), f \circ h_g(x)) + d(f \circ h_g(x), f(x))\} \\ &\leqslant K_4 \{K_3\|s\|_1 + K_2 d(h_g(x), x)\} \leqslant K_4 \{K_3 + K_2 K_1\}\|s\|_1. \end{aligned}$$

Since f preserves Λ, for small $\|s\|_1$, $d(p \circ g \circ h_g \circ f^{-1}(x), x) \leqslant \bar{K}\|s\|_1$. Let $a(x) = p \circ g \circ h_g \circ f^{-1}(x)$. So, for small $\|s\|_1$,

$$d(a(x), x) \leqslant \bar{K}\|s\|_1. \tag{2.14}$$

Towards our goal of verifying (2.10), note that by (2.12)

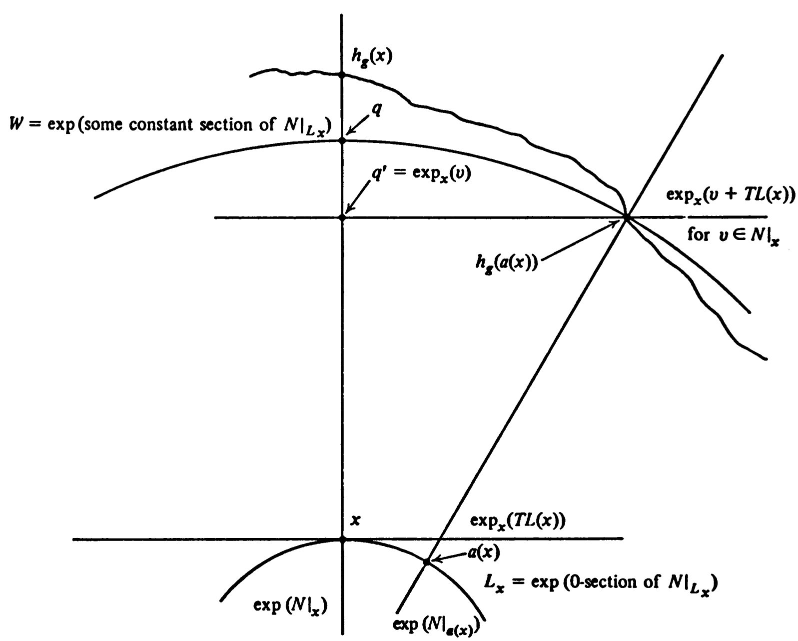
$$\begin{aligned} C(s)(x) &= (\omega_{\exp})^{-1}(g \circ h_g \circ f^{-1}(x)) - (\omega_{\exp})^{-1}(h_g(x)) \\ &= (\omega_{\exp})^{-1}(h_g \circ p \circ g \circ h_g \circ f^{-1}(x)) - (\omega_{\exp})^{-1}(h_g(x)) \\ &= (\omega_{\exp})^{-1}(h_g(a(x))) - (\omega_{\exp})^{-1}(h_g(x)) \end{aligned}$$

for small $\|s\|_1$. Thus, to complete the proof of (2.4), we need only verify that, as $\|s\|_1 \longrightarrow 0$,

$$\|P_N \circ \exp_x^{-1} \circ h_g(a(x)) - P_N \circ \exp_x^{-1}(h_g(x))\|/\|s\|_1 \longrightarrow 0. \tag{2.15}$$

But this looks easy. As f is differentiably L-stable, $d_{1,0}(h_g, i) \longrightarrow 0$ as $\|s\|_1 \longrightarrow 0$. Thus, h_g is locally lipschitz at x with lipschitz constant $K_s \longrightarrow 0$ as $\|s\|_1 \longrightarrow 0$ and, in (2.15) where we are looking at normal components, the numerator should be bounded by $K_s\bar{K}\|s\|_1$. Since $K_s \longrightarrow 0$ as $\|s\|_1 \longrightarrow 0$, we should be through.

Unfortunately, a technical detail remains. Consider the following diagram lying within a small exponential coordinate neighborhood of $x \in M$ over which all relevant bundles are trivial:



The trouble is that h_g is locally lipschitz at x when viewed as a section of $\exp(N)$. That is, to compare $h_g(a(x))$ with $h_g(x)$, one must use exp of constant sections of $N|_{L_x}$. The preceding argument showed that, as $\|s\|_1 \longrightarrow 0$,

$$d(h_g(x), q)/\|s\|_1 \longrightarrow 0. \tag{2.16}$$

But this is not the method prescribed by (2.15) for bringing $h_g(a(x))$ back to $\exp(N|_x)$. Instead of projecting along exp of constant sections of $N|_{L_x}$, one must project along curves of the form $\exp_x(v + TL(x))$ where $v \in N|_x$. This amounts to projecting along $TL(x)$ and results in the point $q' = \exp_x(v)$. So to verify (2.15), one must show that, as $\|s\|_1 \longrightarrow 0$, $d(h_g(x), q')/\|s\|_1 \longrightarrow 0$. By (2.16) it will suffice to show that as $\|s\|_1 \longrightarrow 0$, $d(q, q')/\|s\|_1 \longrightarrow 0$.

Fortunately, this difficulty is easier to alleviate than it was to describe. At x the two different "parallel translations" are tangent (i.e. $\exp_x(0 + TL(x))$ is tangent to $L_x = \exp(0\text{-section of } N|_{L_x})$). Thus they are almost tangent at nearby points since everything in sight is C^1. In particular, for small $\|s\|_1$, $\exp_x(v + TL(x))$ is nearly tangent to W (see preceding diagram) at $h_g(a(x))$. Using the mean value theorem one now gets that either of the two curves is locally lipschitz with lipschitz constant $K'_s \longrightarrow 0$ as $\|s\|_1 \longrightarrow 0$ when the other is used to specify the parallel translation. The above argument together with (2.14) shows that as $\|s\|_1 \longrightarrow 0$,

$$\frac{d(q, q')}{\|s\|_1} \leqslant \frac{K'_s d(a(x), x)}{\|s\|_1} \leqslant \frac{K'_s \bar{K}\|s\|_1}{\|s\|_1} \longrightarrow 0.$$

Consider the following infinitesimal substitute for absolute L-stability. A strongly L-stable diffeomorphism satisfies *condition* B if there exists K such that, for all $s \in C^1(TM, N)$, $\|D\theta(0)(s)\|_0 \leqslant K\|s\|_0$. Since (2.6) is the only use of absolute stability in proving (1.1), we see

(2.17) THEOREM. *Absolute stability may be replaced by condition* B *in* (1.1). *Moreover, absolute stability implies condition* B.

3. **An application to stability of vectorfields.** Let X be a C^r tangent vectorfield on M $(r \geqslant 1)$ and let $\Psi = \Psi^X = \{\Psi_t\}$ be its associated flow. Ψ is called an *automorphism of the lamination* L if each Ψ_t is. A point $p \in M$ is called a *wandering point of* Ψ if there is a neighborhood U of p such that for some t_0, $\Psi_t(U) \cap U = \varnothing$ for all $|t| \geqslant t_0$.

$$\Omega = \Omega(X) = \Omega(\Psi) = \{p \in M | p \text{ is not a wandering point of } \Psi\}.$$

The orbits of Ψ laminate two complementary parts of Ω: F, the zeros of X, becomes 0-laminated and $\Omega - F$ becomes 1-laminated.

Let

$$\begin{array}{ccc} E & \xrightarrow{G} & E \\ \downarrow & & \downarrow \\ B & \xrightarrow{g} & B \end{array}$$

be a Banach bundle isomorphism and let $E^c \longrightarrow \Lambda$ be a (G, g)-invariant Banach subbundle. Following [7], (G, g) is called *normally hyperbolic at* E^c if $E|_\Lambda$ can be split into continuous G-invariant subbundles $E|_\Lambda = E^s \oplus E^c \oplus E^u$ and there is a Finsler on E and constants C, $\epsilon > 0$, and $0 < a, b < 1$ such that for $n > 0$, $\|G^n v\| \leqslant Ca^n \|v\|$ if $v \in E^s$, $\|G^{-n} v\| \leqslant Cb^n \|v\|$ if $v \in E^u$, and for $v \in E^c$, $\|G^n v\| \geqslant C(a + \epsilon)^n \|v\|$ and $\|G^{-n} v\| \geqslant C(b + \epsilon)^n \|v\|$.

If $f \in \mathrm{Diff}^1(M)$ and Ψ are automorphisms of the lamination L, f is called *normally hyperbolic at* L (or *at* Λ) if (Tf, f) is normally hyperbolic at the subbundle TL, and such a Ψ is called *normally hyperbolic at* L (or *at* Λ) if, for some (and hence all, [7]) $t_0 \neq 0$, Ψ_{t_0} is. The usual definition of Axiom A$'$ does not use normal hyperbolicity (see [16]) but in this paper the following equivalent definition will be adopted: X is said to satisfy *Axiom* A$'$a if $\Omega - F$ is compact and Ψ is normally hyperbolic at both F and at $\Omega - F$. X is said to satisfy *Axiom* A$'$b if the closed orbits are dense in $\Omega - F$.

The C^r topology on the space of flows is defined by declaring that the sequence Ψ^j converges to Ψ if the sequence $\Psi^j|_{M \times [0,1]}$ converges to $\Psi|_{M \times [0,1]}$ uniformly in the C^r sense. Two flows Ψ and Ψ' are Ω-conjugate if there is a homeomorphism from $\Omega(\Psi)$ onto $\Omega(\Psi')$ sending sensed orbits onto sensed orbits. A vectorfield X is called Ω-stable if Ψ^X is Ω-conjugate to every flow C^1 near it.

(3.1) Ω-STABILITY THEOREM (FOR VECTORFIELDS) ([13], [16]). *If X satisfies Axiom* A$'$ *and no cycles, then it is Ω-stable.*

No cycles is a global assumption on X the details of which are not needed in this paper. Applying (1.1) and utilizing techniques of [4], one gets the following partial converse to (3.1).

(3.2) THEOREM. *Let X be an Ω-stable C^1 vectorfield on a closed smooth manifold. If $(\Omega - F, L)$ is a compact laminated set and there is a $t_0 \neq 0$ such that $\Psi^X_{t_0}$ is absolutely and differentiably L-stable, then X satisfies Axiom* A$'$ *and no cycles.*

The proof of (3.2) will follow.

(3.3) REMARK. John Franks has told me that I need not assume that $\Omega - F$ is compact.

The assumption that X is Ω-stable is only used to deal with F.

(3.4) COROLLARY. *Let X be a C^1 nowhere zero vectorfield. If $\Psi^X_{t_0}$ is absolutely and differentiably (Ω, L)-stable, then X satisfies Axiom* A$'$ *and no cycles.*

Using (2.17) we get

(3.5) THEOREM. *Absolute stability may be replaced by condition* B *in* (3.2) *and* (3.4).

PROOF OF (3.2). Using Pugh's density theorem [12] and work of Palis [11], (3.2) is reduced to verifying Axiom A$'$a given the additional assumption that X satisfies Axiom A$'$b. By [8] and [9] each singularity of X is hyperbolic. Thus they are finite in number. But for a 0-laminated set, normal hyperbolicity is the existence of a and C, uniform hyperbolicity constants valid for each point of the set. Thus Ψ^X is normally hyperbolic at F. One hypothesis of (3.2) is that $\Omega - F$ is compact so we need only verify normal hyperbolicity at $\Omega - F$.

Let $f = \Psi^X_{t_0}$ (t_0 as specified in (3.2)) and let $N \longrightarrow \Omega - F$ and $\overline{f_\#}$ be as in §2. If $\mathrm{Sp}(\overline{f_\#})$, the spectrum of $\overline{f_\#}$: $C^0(N) \longrightarrow C^0(N)$, misses the unit circle, $C^0(N)$ splits into expanding and contracting subspaces. By [7] and [22] this induces the required splitting of $TM|_{\Omega - F}$. In fact we need not worry about the whole circle S^1.

(3.6) LEMMA. *If* $1 \notin \mathrm{Sp}(\overline{f_\#})$, *neither is any* $z \in S^1$ (*cf.* [4, *Lemma* 1]).

PROOF OF (3.6). To be precise we must complexify. Let

$$B = \{s + is' \mid s, s' \in C^0(N)\}$$

considered as a complex vector space with norm

$$\|s + is'\| = \sup_{a^2+b^2=1} (\|as + bs'\|^2 + \|bs + as'\|^2)^{1/2}.$$

Let $L: B \longrightarrow B$ be defined by $L(s + is') = \overline{f_\#}(s) + i\overline{f_\#}(s')$. Clearly $1 \notin \mathrm{Sp}(L)$ so there exists $\epsilon > 0$ such that

(3.7) For all $w \in B$, $\|(\mathrm{id} - L)\| \geqslant 4\epsilon$.

Choose n satisfying $1/n < \epsilon$. We may decompose $\Omega - F$ as the disjoint union $\bigcup L_x$ of appropriate lamina L_x, where each laminum is a nontrivial orbit of the flow Ψ. By continuity, either L_x is wholly isolated in $\Omega - F$ or each point of L_x is a limit point of $(\Omega - F) - L_x$. Let $\Lambda_0 \subset \Omega - F$ be those x's such that L_x is an isolated closed f-periodic orbit with period $\leqslant 2n + 1$.

(3.8) ASSERTION. *There are only finitely many closed f-periodic orbits of a given period* p.

PROOF OF (3.8). Since the orbits are normally hyperbolic fixed circles for f^p (see [8] and [9]), a simple argument, using the Poincaré map, shows that they are isolated from each other. Now, assume that there are infinitely many, and choose a countable subset $\{L_{x_i}\}$. By compactness, we may assume that $x_i \longrightarrow y \in \Omega - F$. Since $\Psi_t(x_i) \longrightarrow \Psi_t(y)$ and $\Psi_p(x_i) = x_i$, L_y is a closed orbit having period a divisor of p. By the argument alluded to above, L_y would be isolated from the L_{x_i}'s, which is impossible.

Let $\Lambda = (\Omega - F) - \Lambda_0$. Both Λ_0 and Λ are closed f-invariant sets and we may write Λ as the disjoint union, $\bigcup L_x$. Each L_x falls into precisely one of the following classes.

(1) L_x is not a closed orbit.

(2) L_x is a closed but not f-periodic orbit.

(3) L_x is f-periodic with period $> 2n + 1$.

(4) L_x is f-periodic with period $\leqslant 2n + 1$.

(3.9) ASSERTION. *$\Lambda'' = \{x | L_x$ is in class (2) or class (3)$\}$ is dense in Λ.*

PROOF OF (3.9). Choose $L_{x'}$ in class (4). Since $L_{x'} \subset \Lambda$, it is not isolated. So $L_{x'} \subset \operatorname{clos}((\Omega - F) - L_{x'})$. In fact $L_{x'} \subset \operatorname{clos}(\Lambda - L_{x'})$. Let $\Lambda' \subset \Lambda$ be those L_x not in class (1). By Axiom A′b, $\operatorname{clos}(\Lambda') = \Lambda$.

(3.10) CLAIM. $L_{x'} \subset \operatorname{clos}(\Lambda' - L_{x'})$.

From this claim and the finiteness of class (4) (by (3.8)), (3.9) follows.

PROOF OF (3.10). For all m choose $x_m \in \Lambda - L_{x'}$ with $d(x_m, x') \leqslant 1/m$. If $x_m \in \Lambda'$, let $y_m = x_m$; otherwise, use Axiom A′b to choose $y_m \in \Lambda'$ with $d(y_m, x_m) \leqslant 1/m$. By compactness of $L_{x'}$, we may choose $y_m \notin L_{x'}$. Thus $x' \in \operatorname{clos}(\Lambda' - L_{x'})$ and, therefore, so is all of $L_{x'}$.

Returning to (3.6), let $B'' = \{s \in B | \, s|_{\Lambda_0} = 0\}$ and choose $z \in S^1$.

(3.11) ASSERTION. *For all $s \in B''$, $\|(L - z \text{ id})(s)\| \geqslant \epsilon \|s\|$.*

PROOF OF (3.11). If not, there exists $\|(L - z \text{ id})(s)\| < \epsilon \|s\|$ with $\|s\| = 1$. By (3.9) we can find $x \in \Lambda''$ with $\|s(x)\| > \frac{1}{2}$. The same argument given in [4] applies here since we have a compact orbit not of low period. This argument would result in a contradicition to (3.7) and completes the proof of (3.11).

Let $B' = \{s \in B | \, s|_{\Lambda} = 0\}$ and recall that Λ_0 consists of a finite number of normally hyperbolic f-invariant circles. Using the alternate characterization of normal hyperbolicity given in [7], we can find $\epsilon'' > 0$ such that for all $s \in B'$ and all $z \in S^1$, $\|(L - z \text{ id})(s)\| \geqslant \epsilon'' \|s\|$. Since Λ and Λ_0 are disjoint closed sets, $B = B'' \oplus B'$. Thus, if we let $\epsilon' = \min(\epsilon, \epsilon'')$,

$$\|(L - z \text{ id})(s)\| \geqslant \epsilon' \|s\| \quad \text{for all } z \in S^1 \text{ and all } s \in B. \tag{3.12}$$

From (3.12) one can show, precisely as Franks [4] does, that $\operatorname{Sp}(L) \cap S^1 = \emptyset$ which concludes the proof of (3.6).

Lemma (3.6) reduces Theorem (3.2) to showing that $(\text{id} - \overline{f_\#})$ is an isomorphism. By the open mapping theorem, we need only show that it is bijective. Surjectivity is just Theorem (1.1) so we need only show the following.

(3.13) LEMMA. $\text{id} - \overline{f_\#}$ *is injective.*

PROOF OF (3.13). Choose $0 \neq s \in C^0(N)$. By Axiom A′b, there exists $x \in \Omega$ with $s(x) \neq 0$ and L_x a closed orbit. By [8] and [9], f is normally hyperbolic at L_x so, by [7],

$$\mathrm{Sp}(\overline{f_\#}|_{C^0(N|_{L_x})}) \cap S^1 = \varnothing.$$

In particular $(\overline{f_\#} - \mathrm{id})|_{C^0(N|_{L_x})}$ is bijective. Since $s|_{L_x} \neq 0$,

$$(\overline{f_\#} - \mathrm{id})|_{C^0(N|_{L_x})}(s|_{L_x}) \neq 0$$

and therefore $(\overline{f_\#} - \mathrm{id})(s) \neq 0$.

4. Possible improvements to Theorem (3.2). A major improvement would be to omit the assumption that $\Psi^X_{t_0}$ is $(\Omega - F, L)$-stable as a diffeomorphism and assume only that $\Psi^X_{t_0}$ and $\Psi^{X+s}_{t_0}$ are conjugate for C^1 small perturbations s. This is more in the spirit of flows and is weaker since there are diffeomorphisms C^1 near f which do not imbed in a flow. All of §3 still works with this weaker assumption, so one would need only show that $(\mathrm{id} - (\Psi^X_{t_0})_\#)$ is surjective.

Another improvement would be to show that the hypotheses of (3.2) are necessary. The most significant result in this direction is (3.1) which shows that Ω-stability is necessary. Actually, the proof of (3.1) shows more: The key step in the proof is an application of the invariant lamination theorem which, in fact, shows that strong $(\Omega - F, L)$-stability as well as condition (2) of differentiable $(\Omega - F, L)$-stability are necessary. Mike Shub has shown me an alternate proof of the invariant lamination theorem which seems to imply that part of condition (1) is necessary: The map h is lipschitz. Of course the assumption that $\Omega - F$ is compact is necessary for Axiom A′a.

Recall from §1 Smale's conjecture that Axiom A′ and no cycles are equivalent to Ω-stability. Removing any of the other hypotheses of (3.2) brings us closer to that goal. As previously mentioned, Franks believes we can drop the assumption that $\Omega - F$ is compact.

REFERENCES

1. R. Abraham and J. Robbin, *Transversal mappings and flows*, Benjamin, New York, 1967. MR 39 #2181.
2. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966. MR 33 #1824.
3. J. Eells, Jr., *A setting for global analysis*, Bull. Amer. Math. Soc. 72 (1966), 751–807. MR 34 #3590.
4. J. Franks, *Differentiably Ω-stable diffeomorphisms*, Topology 11 (1972), 107–113. MR 45 #6031.
5. J. Guckenheimer, *Absolutely Ω-stable diffeomorphisms*, Topology 11 (1972), 195–197.
6. M. Hirsch, J. Palis, C. Pugh and M. Shub, *Neighborhoods of hyperbolic sets*, Invent. Math. 9 (1969/70), 121–134. MR 41 #7232.

7. M. Hirsch, C. Pugh and M. Shub, *Invariant manifolds*, Lecture Notes in Math., Springer, Berlin and New York (to appear). A partial statement of results with some idea of proof has appeared in Bull. Amer. Math. Soc. **76** (1970), 1015–1019.

8. L. Markus, *Structurally stable differential systems*, Ann. of Math. (2) **73** (1961), 1–19. MR **24** #A2724.

9. ———, *The behavior of the solutions of a differential system near a periodic solution*, Ann. of Math. (2) **72** (1960), 245–266.

10. Z. Nitecki, *Differentiable dynamics. An introduction to the orbit structure of diffeomorphisms*, M.I.T. Press, Cambridge, Mass., 1971.

11. J. Palis, *A note on Ω-stability*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I., 1970, pp. 221–222. MR **42** #5276.

12. C. Pugh, *An improved closing lemma and a general density theorem*, Amer. J. Math. **89** (1967), 1010–1021. MR **37** #2257.

13. C. Pugh and M. Shub, *The Ω-stability theorem for flows*, Invent. Math. **11** (1970), 150–158. MR **44** #4782.

14. C. Robinson (unpublished).

15. M. Shub, *Stability and genericity for diffeomorphisms*, Dynamical Systems, M. M. Peixoto (Editor), Academic Press, New York, 1973.

16. S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817. MR **37** #3598.

17. ———, *The Ω-stability theorem*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I., 1970, pp. 289–297. MR **42** #6852.

18. N. Steenrod, *Topology of fibre bundles*, Princeton Math. Ser., vol. 14, Princeton Univ. Press, Princeton, N. J., 1951. MR **12**, 522.

19. G. Birkhoff, *Dynamical systems*, Amer. Math. Soc. Colloq. Publ., vol. 9, Amer. Math. Soc., Providence, R. I., 1927.

20. D. V. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature*, Trudy Mat. Inst. Steklov. **90** (1967) = Proc. Steklov Inst. Math. **90** (1967). MR **36** #7157; **39** #3527.

21. R. Mañé, *Persistent manifolds are normally hyperbolic* (to appear).

22. J. N. Mather, *Characterizations of Anosov diffeomorphisms*, Nederl. Akad. Wetensch. Proc. Ser. A **71** = Indag. Math. **30** (1968), 479–483. MR **40** #2129.

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